A Brief Survey of Art Gallery Problems in Integer Lattice Systems

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The camera placement problem concerns the placement of a fixed number of point-cameras on the integer lattice of d-tuples of integers in order to maximize their visibility. We survey some of the combinatorial optimization and algorithmic techniques which have been developed in order to study this and other similar problems in the context of lattices and more generally, in combinations of lattice systems and tilings.

1 Introduction

Visibility and illumination problems are among the most appealing and intuitive research topics of combinatorial geometry. In many cases (though not all) their analysis requires nothing more than basic topics from geometry, number theory and graph theory and as such they are very well suited for a wide audience [2]. In recent years there has been particular emphasis on the algorithmic component of visibility problems in polygonal configurations; as such they have come to be studied under the area of "art gallery (watchman) problems" which in turn lies at the intersection of combinatorial and computational geometry [16].

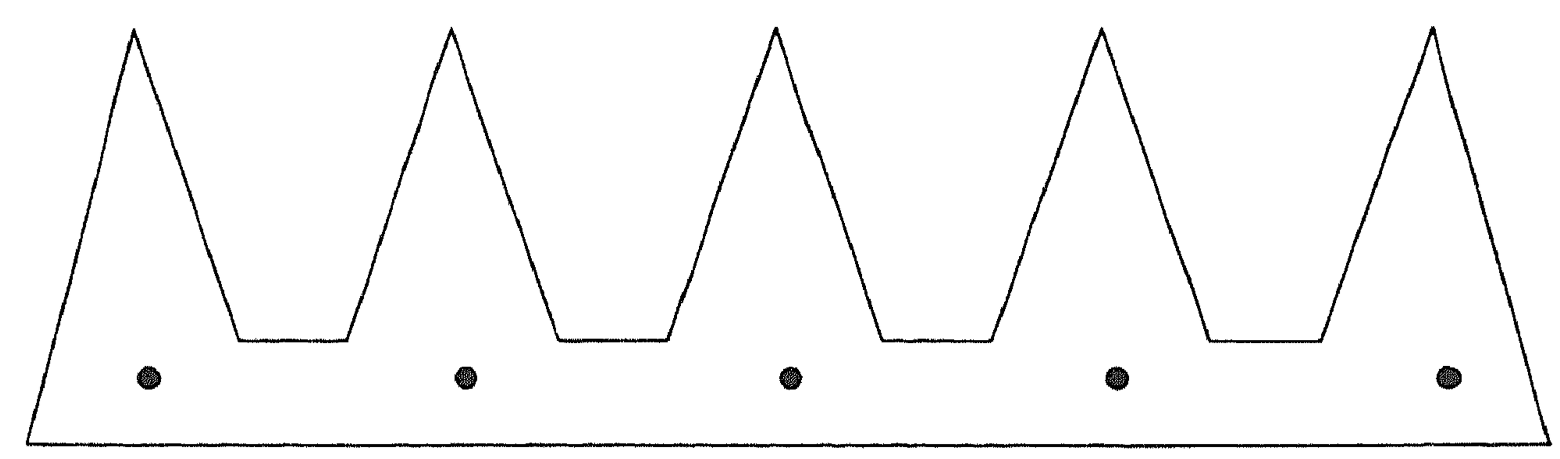

FIGURE 1. $\lfloor n/3 \rfloor$ guards are sufficient and sometimes necessary to cover an n-wall art gallery.

Art gallery problems, theorems and algorithms are so named after the celebrated question first posed by V. Klee in 1973: "What is the minimum number of guards sufficient to cover the interior of an n-wall gallery?" The problem was solved soon thereafter first by Chvátal and subsequently also by Fisk (see Figure 1). Since then art gallery problems have successfully emerged as a research area that stresses complexity and algorithmic aspects of visibility and illumination in configurations comprising "obstacles" and "guards". In fact by creating rather idealized situations the theory succeeds in abstracting the algorithmic essence of many visibility problems (like in partitioning theorems, mobile guard configurations, visibility graphs, etc.) thus significantly facilitating the study of their computational complexity.

In the present paper we focus on visibility with respect to the d-dimensional integer lattice Λ : a point y is *visible* from point x if the line segment from x to y contains no points in Λ, other than x and y. In particular we are interested in the following art gallery problem.

> **The s-camera placement problem:** *Given an integer s, determine a collection S (of camera locations) contained in Λ and of cardinality s, such that the density of lattice points which are visible from at least one point of S is as large as possible.*

1.1 Some definitions

Before providing an outline of the main themes of investigation we remind the reader of some basic definitions and simple facts. By Λ we denote the d dimensional integer lattice consisting of d-tuples of integers and by Λ_n the set of lattice points in Λ having absolute value $\leq n/2$. Very important for our subsequent optimization analysis is the notion of density of a set of lattice points. By *density* of the set $X \subseteq \Lambda$ we understand the limit (if it exists) of the ratio $|X \cap \Lambda_n|/n^d$, as n goes to infinity.

Let $\mathcal{P} = \{2, 3, 5, \ldots\}$ be the set of prime numbers, p ranges over $\mathcal{P}$. Two lattice points x and y are *p-visible* if they are distinct modulo p; two points x, y which are p-visible for all primes $p \in \mathcal{P}$ are visible in the geometric sense, i.e. the line segment joining x and y avoids all the lattice points but x, y (see Figure 2). For all $X \subseteq \Lambda$, X/p denotes the quotient set of X by the relation of equality modulo p. An element of X/p is also called a *coset* of X/p.

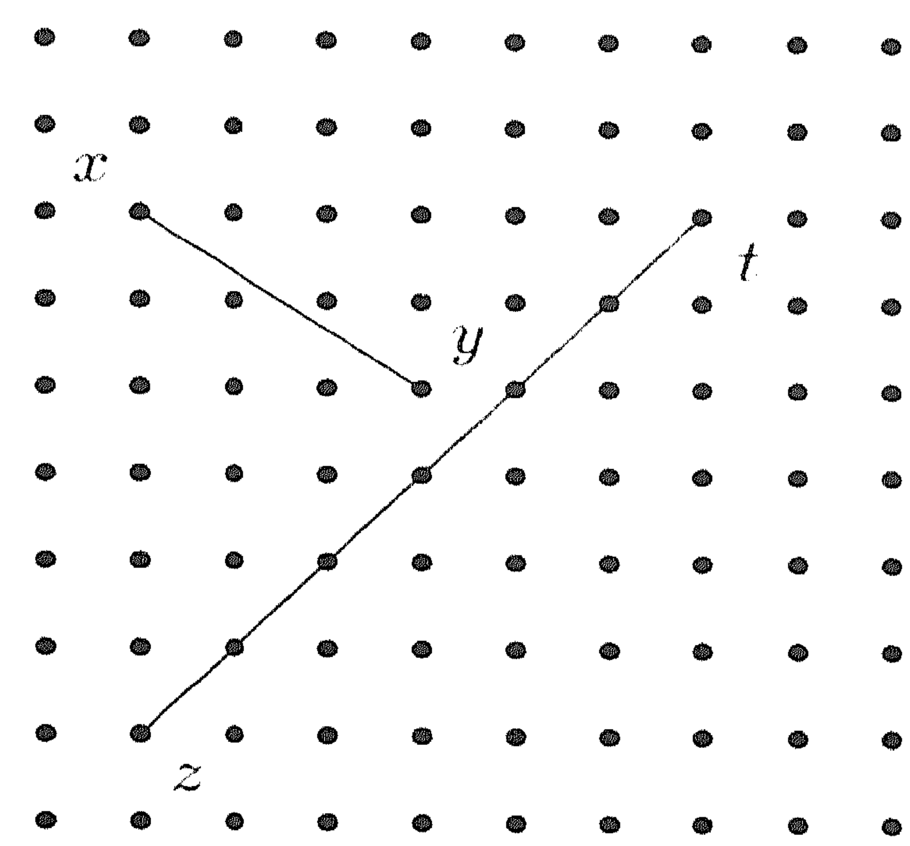

FIGURE 2. Points x and y are visible; points z and t are p-visible for $p \neq 2, 3$.

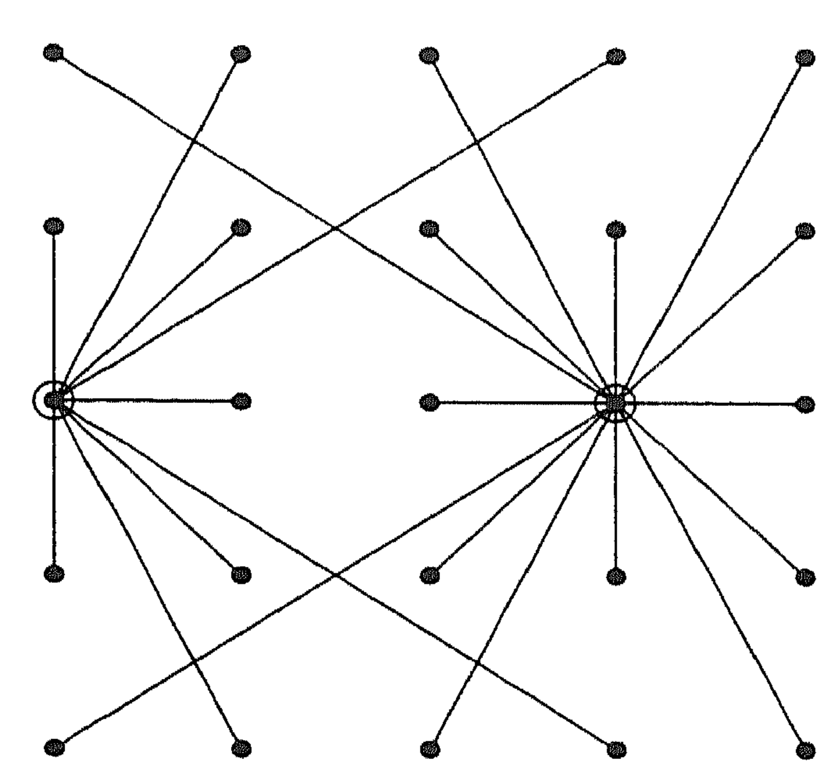

FIGURE 3. Two guards are enough to cover lattice of pairs of integers $\leq 5/2$.

1.2 Related literature

Interesting visibility problems have been studied on integer lattices [5, 10]. Of these we single out two which are relevant for our study.

Rumsey [20] shows that for any set S of lattice points, the density of the set of lattice points visible from each point of S is given by the infinite product

$$\prod_{p \in \mathcal{P}} \left(1 - \frac{|S/p|}{p^d}\right). \tag{1}$$

(In fact, Rumsey gives a characterization of the sets S for which the density formula (1) is true.) The above formula was previously obtained by G. Leujeune Dirichlet for the case $|S| = 1$ ("the probability that d integers chosen at random are relatively prime is $1/\zeta(d)$", where $\zeta(z) = \sum_{n \geq 1} n^{-z}$, $|z| > 1$, denotes the Riemann zeta function, [11, page 324]) and by Rearick [18, 19] for the case where $|S| = 2$ and the points of S are pairwise visible.

An interesting (and in general still open) art gallery problem was posed by Moser [15] in 1966: given a set P of points in the plane how many guards located at points of P are needed to see the unguarded points of P? Abbott [1] studies the case $P = \Lambda_n$ and shows that the minimum number $f(n)$ of guards which are necessary in order to see all the points of Λ_n (see Figure 3) verifies the inequalities

$$\frac{\ln n}{2 \ln \ln n} < f(n) < 4 \ln n.$$

The lower bound result follows by applying the Chinese remainder theorem and the Prime number theorem. For the upper bound Abbott constructs recursively a sequence $x_1, x_2, \ldots, x_k$ such that for each i, x_{i+1} is a point x in the set Λ_n for which the set-theoretic difference $V_n(x) \setminus (V_n(x_1) \cup \cdots \cup V_n(x_i))$, where $V_n(x)$ is the set of points of Λ_n visible from x, is of maximal size and shows that $k = O(\ln n)$ iterations of this procedure suffice in order to cover all the vertices of the lattice. His method however gives no "qualitative" information on the location of these points on the lattice. Nevertheless, he also shows using work of Erdös [4] that there exists a constant $\alpha > 0$ such that, for $d = 2$, every point of the lattice Λ_n is visible from the set $\{ (1,0)\} \cup \{(0,j) \mid j = 0, 1, \ldots, k \}$, where $k = O(\ln^\alpha n)$. It is straightforward to see that his methods can easily be extended in order to yield similar results for the d-dimensional lattice Λ_n.

2 Camera Placement Problem

The *camera placement problem* in multidimensional lattices is the following. We are given s cameras $C_1, \ldots, C_s$ which are supposed to be located on the points of the d-dimensional lattice Λ. We are interested in determining a set $S = \{ A_1, \ldots, A_s \}$ of positions (lattice points) for these cameras in such a way that if camera C_i is positioned at location A_i, for $i = 1, \ldots, s$, then the density of the lattice points visible by at least one of the cameras is maximized. More formally, we want to find conditions on the set S of possible camera locations so that the quantity

$$u(S) := \sum_{E \subseteq S, E \neq \emptyset} (-1)^{|E|+1} \prod_{p \in \mathcal{P}} \left(1 - \frac{|E/p|}{p^d}\right), \tag{2}$$

which is obtained from (1) using the principle of inclusion/exclusion is maximized. Configurations which attain the optimal density will be called *optimal.*

Example 1 *Figure 4 displays two* 5*-camera configurations:* $\{1, 2, 3, 4, 5\}$ *and* $\{a, b, c, d, e\}$*. One of the questions we will study in this paper is which of these camera configurations has optimal visibility.*

The camera placement problem can be thought of as a "qualitative" version of Abbott's problem already stated in the introduction. Despite the fact that Abbott's (and hence Moser's) question still remains open we expect that our investigations will also contribute to a better understanding of this problem. Indeed, since the density $u(S)$ depends only on the relation of p-visibility on the cameras, we expect to deduce some qualitative information on the locations of the cameras that achieve the $O(\log n)$ upper bound of Abbott's theorem.

In the present paper we outline some of the optimization results developed for the solution of the camera placement problem. No proofs will be given here, but the reader interested in more detailed accounts is advised to consult [17, 12, 13].

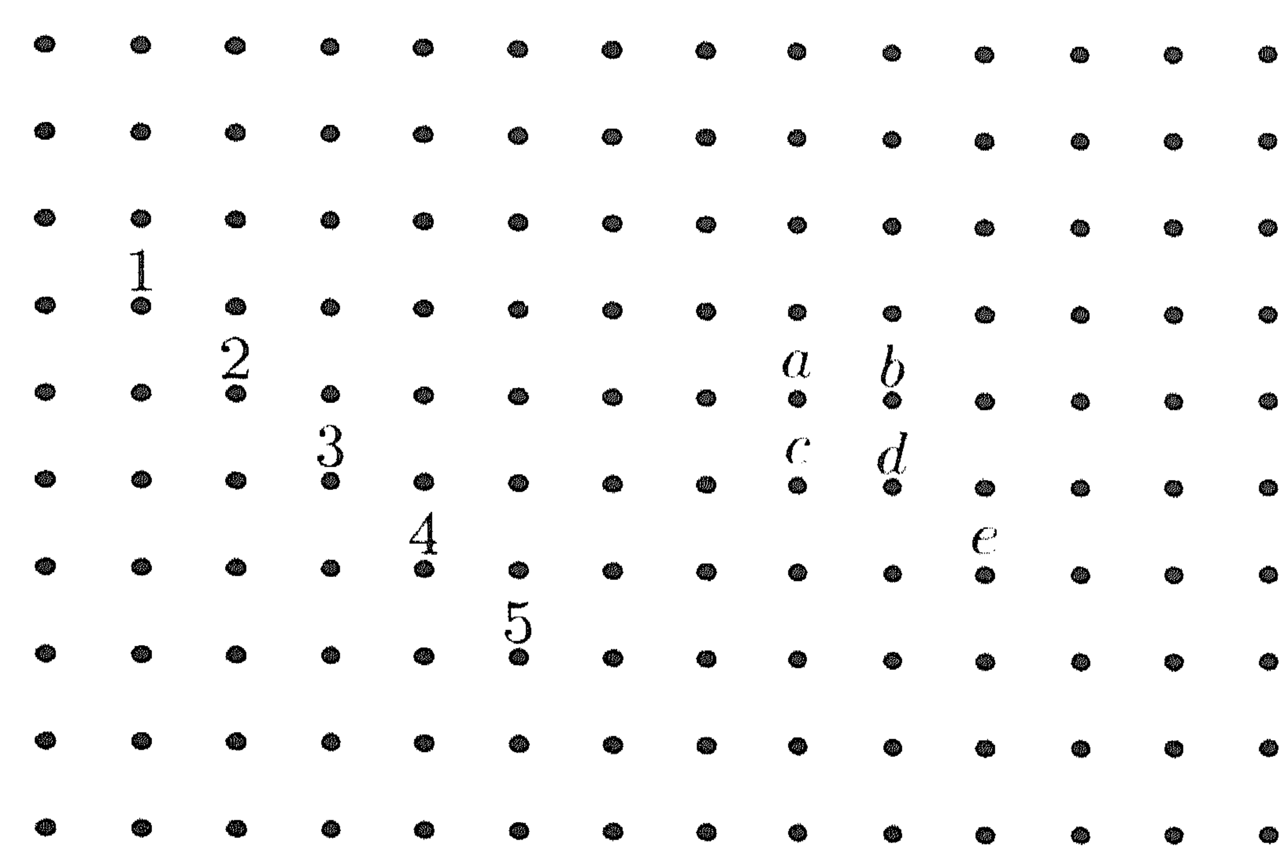

FIGURE 4. Two 5-camera configurations $\{1, 2, 3, 4, 5\}$ and $\{a, b, c, d, e\}$.

2.1 Reduction to a non-linear integer optimization problem

The difficulty of the optimization problem previously stated is due not only to the way we specify and manipulate the locations of the cameras, but also on the formulation of $u(S)$ as an alternating sum in identity (2). In the sequel it will be necessary to reformulate the problem as a non-linear integer optimization problem. To accomplish this we introduce, for $\mathcal{Q}$ subset of $\mathcal{P}$, the *$\mathcal{Q}$-density* of the configuration S as the density, denoted by $u(\mathcal{Q}, S)$, of the set of lattice points p-visible for all $p \in \mathcal{Q}$ from at least one point of the configuration S. It can be shown that for any prime p the $\mathcal{Q}$-density of the configuration S is the mean of the $\mathcal{Q} \setminus \{p\}$-density of the p^d sub-configurations $S \setminus c$ where c ranges over the cosets of Λ/p

$$p^d\, u(\mathcal{Q}, S) = \sum_{c \in \Lambda/p} u(\mathcal{Q} \setminus \{p\}, S \setminus c). \tag{3}$$

As a first consequence of this expression we get that a necessary optimality condition is that

$$\text{For all } p \in \mathcal{P},\ |S/p| = \min\{s, p^d\}. \tag{4}$$

This means that the cameras of an optimal configuration must be located in different cosets of Λ/p, for $p^d \geq s$, in such a way that condition (4) is satisfied.

The difficulty of the problem is now to determine the "optimal" repartition of the cameras in the cosets of $\Lambda/2, \Lambda/3, \ldots, \Lambda/p_r$ where $p_1 = 2, p_2 = 3, \ldots, p_r$ is the (finite) increasing sequence of prime numbers less than $s^{1/d}$. To each configuration S (satisfying (4)) we associate the family of integers (a_c) indexed by the elements $c = (c_1, \ldots, c_r) \in \mathcal{C} := \Lambda/2 \times \cdots \times \Lambda/p_r$ defined by

$$a_c = |S \cap c_1 \cap c_2 \cap \ldots \cap c_r|.$$

It turns out that this family of numbers determines the density $u(S)$. Conversely, it can be shown that given a family of numbers $(a_c)_{c \in \mathcal{C}}$ there exists a configuration S of $s = \sum_c a_c$ points, which satisfies (4), and to which the family (a_c) is associated by the above described procedure.

To clarify these definitions we give in the sequel two examples.

EXAMPLE 2 *Consider the two configurations of Figure 4. For* $\{1,2,3,4,5\}$, *the equivalence classes modulo* 2 *are* $\{2,4\},\{1,3,5\}$, *while the equivalence classes modulo* 3 *are* $\{1,4\}$, $\{2,5\}$, $\{3\}$. *For* $p \geq 5$, *the equivalence classes modulo* p *are the singletons. For* $\{a,b,c,d,e\}$, *the equivalence classes modulo* 2 *are* $\{a,e\},\{b\},\{c\},\{d\}$, *while for* $p \geq 3$, *the equivalence classes modulo* p *are the singletons.*

EXAMPLE 3 *In the configuration* $\{1,2,3,4,5,6,7,8,9\}$ *of Figure 5 the equivalence classes modulo* 2 *are* $\{1,7\},\{4,9\},\{3,5\},\{2,6,8\}$, *while for* $p \geq 3$, *the equivalence classes modulo* p *are the singletons.*

Equipped with this new way of specifying a configuration of cameras we give now a new expression for the function $u(S)$ to be maximized. We introduce the *reduced density function*, defined on the subsets E of S by

$$u'(E) := u(\mathcal{P} \setminus \{p_1, \dots, p_r\}, E),$$

and the family of *reduced configurations* $\mathcal{B}_c \subseteq S$ defined by

$$\mathcal{B}_c = S \setminus \bigcup_{i=1}^{i=r} c_i.$$

Then by a repeated application of (3) we get that the visibility of the configuration S is the mean of the $\mathcal{P} \setminus \{p_1, \dots, p_r\}$-density of the $m^d := p_1^d \dots p_r^d$ reduced configurations, i.e.

$$m^d\, u(S) = \sum_{c \in \mathcal{C}} u'(\mathcal{B}_c). \tag{5}$$

It turns out, under the assumption that the configuration S satisfies (4), that the reduced density function $u'(E)$ depends only on the size $|E|$ of the set E and it can be verified that $u'(e) := u'(E)$, where $e = |E|$, is absolutely monotone. Using the standard notation of the calculus of finite differences

$$\Delta^1 f(x) = f(x+1) - f(x), \quad \Delta^{n+1} f = \Delta^1(\Delta^n f)$$

this means that $(-1)^{n+1}\Delta^n u'(e) > 0$ for all integers $n \geq 1$. Furthermore, the cardinal b_c of the reduced configuration $\mathcal{B}_c$ can be expressed as a function of the family of integers a_c by the relation

$$b_c = \sum_{h(c,c')=r} a_{c'}$$

where $h(c,c')$, the *Hamming distance*, is defined as the number of i such that $c_i \neq c'_i$. We have reformulated our optimization problem in terms of the following non-linear integer optimization problem

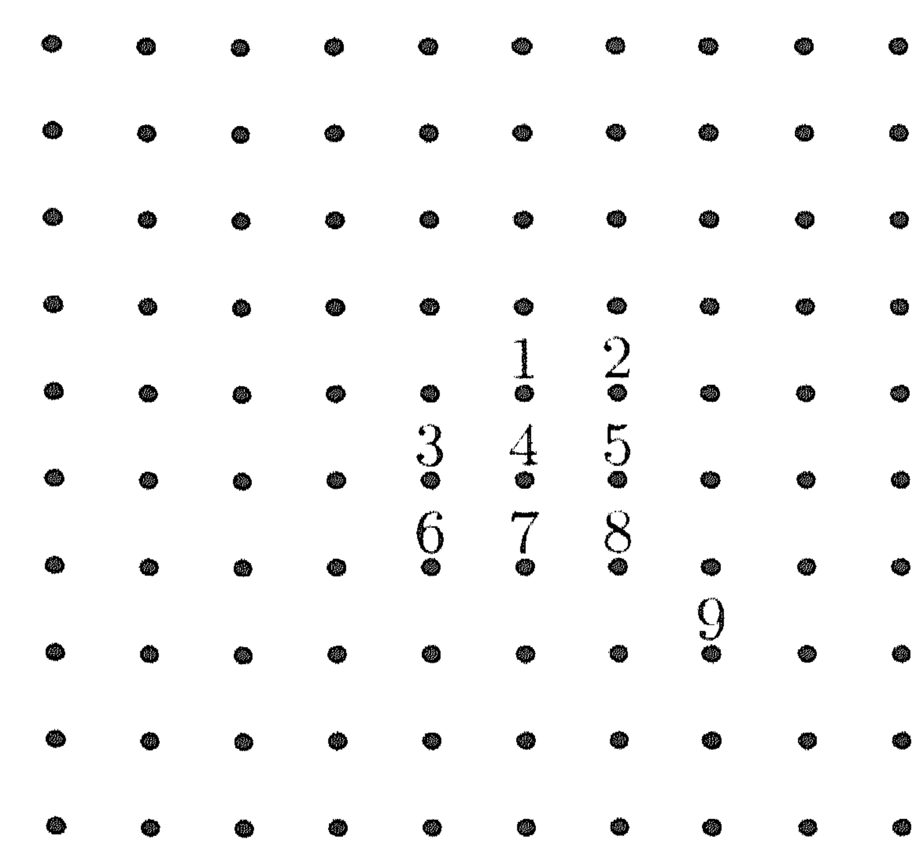

FIGURE 5. For each $s \leq 9$ the configuration $\{1, \ldots, s\}$ is optimal.

$$\max \left\{ \sum_{I \in \mathcal{I}} u'(b_I) \mid b_I = \sum_{\substack{h(I,J)=r \\ J \in \mathcal{I}}} a_J,\ \sum_{I \in \mathcal{I}} a_I = s,\ a_I \in \mathbb{N} \right\} \tag{6}$$

where $\mathcal{I} = [1..p_1^d] \times \cdots \times [1..p_r^d]$ is a set of multi-indices, h is the Hamming distance and the function u' is an absolutely monotone function.

The concavity of u' ($\Delta^2 u' < 0$) and the fact that the terms b_I sum to a constant ($= s \prod_{i=1}^{i=r} (p_i^d - 1)$) suggest that for an optimal configuration the numbers b_I must "differ from each other by a minimum amount". A classical measure of this deviation is the sum $\sum_I b_I^2$ of the squares of numbers b_I, which we will call the *variance* of the configuration S.

The previous considerations enable us to conjecture that an optimal configuration must be of minimal variance. All our subsequent considerations are guided by this conjecture which we confirm for the case of "almost all" configurations of $s \leq 5^d$ cameras. Moreover we have the following characterization.

THEOREM 4 *The variance of a configuration S is minimal if and only if for every square free integer n and every c and $c' \in \Lambda/n$ the cardinals of $c \cap S$ and $c' \cap S$ differ by at most one.*

In other words, configurations of minimal variance are precisely the ones whose cameras are "equally distributed" with respect to the equivalence classes modulo n, for n square free integer. In addition it is easy to see that configurations of minimal variance must satisfy condition (4).

2.2 Optimization for $s \leq 5^d$ cameras

The previous transformations make it possible to give elegant characterizations of optimal configurations of $s \leq 3^d$ cameras.

THEOREM 5 *A configuration S of size $\leq 3^d$ is optimal if and only if its variance is minimal.*

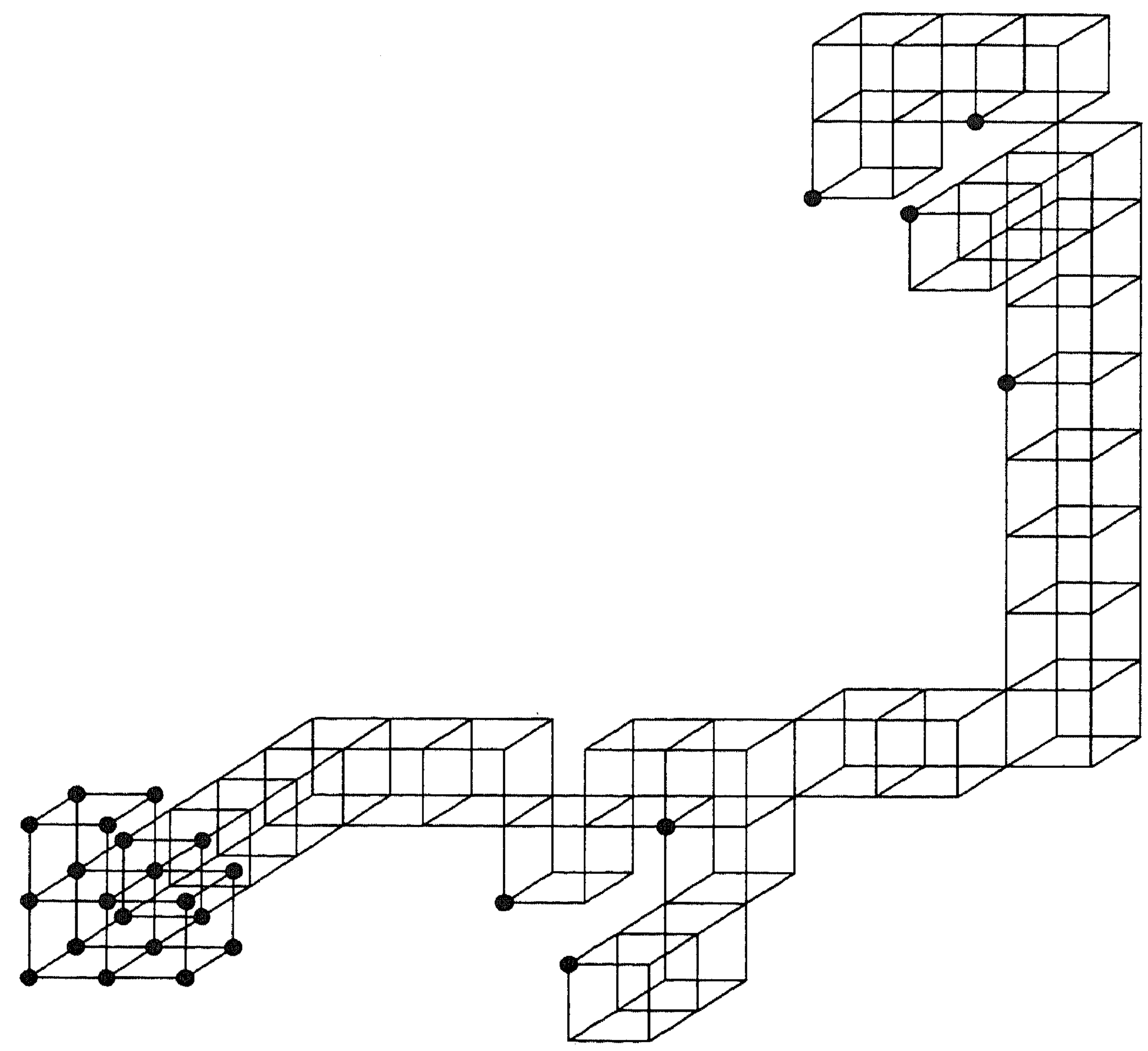

FIGURE 6. Optimal Configuration of 27 cameras in dimension 3

Thus, a configuration S of size $\leq 2^d$ is optimal if and only if its cameras are pairwise p-visible for all primes p, while a configuration S of size $\leq 3^d$ is optimal if and only if its cameras are pairwise p-visible for all primes $p \geq 3$, and for all $c, c' \in \Lambda/2$ $||S \cap c| - |S \cap c'|| \leq 1$.

EXAMPLE 6 *Again, consider the configurations displayed in Figure 4 and mentioned in Example 2. The first configuration* $\{1, 2, 3, 4, 5\}$ *is not optimal because cameras* $1, 4$ *are not* 3*-visible. On the other hand, configuration* $\{a, b, c, d, e\}$ *is optimal.*

EXAMPLE 7 *It is easy to check that all nine configurations* $\{1, \ldots, s\}$, $1 \leq s \leq 9$, *depicted in Figure 5 are optimal. Notice that this example in conjunction with the five camera configuration* $\{a, b, c, d, e\}$, *depicted in Figure 4, shows that optimal configurations are not uniquely determined.*

EXAMPLE 8 *A* 27*-camera optimal configuration in three dimensional space is depicted in Figure 6.*

For $3^d < s \leq 5^d$ the problem is much more difficult. Let $L_1, \ldots, L_{2^d}$ be the 2^d cosets of $\Lambda/2$, $C_1, \ldots, C_{3^d}$ the 3^d cosets of $\Lambda/3$ and let us use the abbreviations $l_i = |L_i \cap S|$, $c_j = |C_j \cap S|$, $a_{i,j} = |L_i \cap C_j \cap S|$. Now recall that our optimization problem has been transformed to the following one

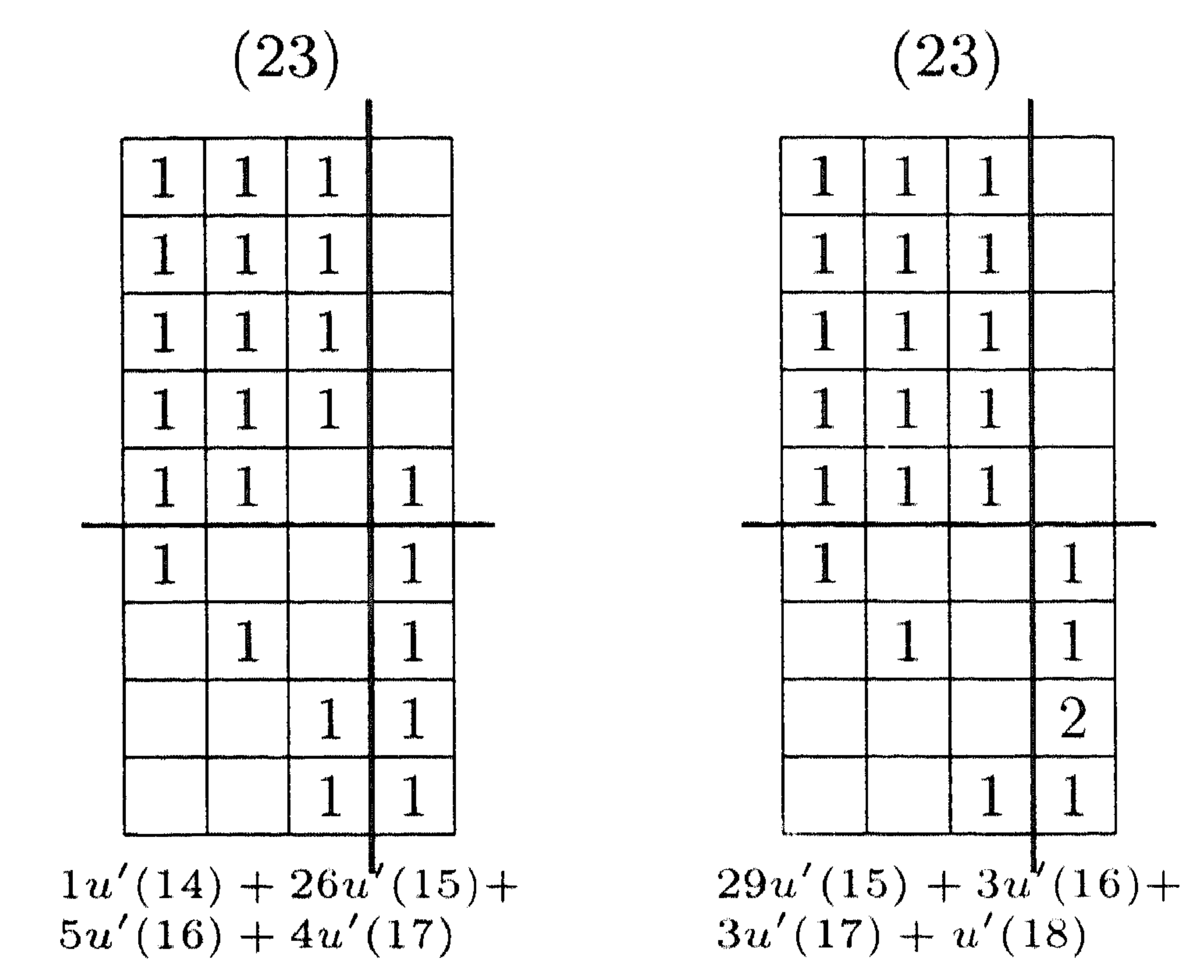

FIGURE 7. The two candidates to optimality when $s = 23$

$$\max \left\{ \sum_{i,j} u'(b_{i,j}) \mid b_{i,j} = \sum_{k \neq i, l \neq j} a_{k,l}, \quad \sum_{i,j} a_{i,j} = s, \quad a_{i,j} \in \mathbb{N} \right\}. \tag{7}$$

Then we can show (and this is not easy [17]) the following conditions on optimal configurations

THEOREM 9 *If S is an optimal configuration then*

$$|l_i - l_j| \leq 1 \quad \textit{and} \quad |c_i - c_j| \leq 1.$$

Furthermore there exists an integer $\delta \leq s$ such that, after any permutation of the indices which insures that the sequences l_i and c_j are decreasing,

$$a_{i,j} = \begin{cases} \delta \quad \textit{or} \quad \delta + 1 & \textit{if } (i,j) > (i_0, j_0) \\ 0 \quad \textit{or} \quad 1 & \textit{otherwise} \end{cases}$$

where $i_0 = s \bmod (2^d)$ and $j_0 = s \bmod (3^d)$.

The above characterization fails, in general, to give the exact value of δ. Nevertheless it can be shown that for almost all values of s (the ratio is at least $(1 - (5/6)^d)$) we have $\delta = 0$. In that case the above constraints are equivalent to the minimality of the variance of the configuration. So, in general, we are forced to conduct a search for a number of configurations which are "candidates" to optimality, one for each value of δ between 0 and s. The following theorem shows that this search is equivalent to a linear integer optimization problem which is solvable in polynomial time.

THEOREM 10 *Optimal configurations among the ones satisfying the constraints expressed in Theorem 9 are characterized by the condition*

$$\sum_{i > i_0, j > j_0} a_{i,j} \textit{ is maximal.}$$

Furthermore this last optimization problem can be solved in time $O(s \log s)$.

For example, in two dimensions there is only one candidate to optimality for every value of $s \leq 5^2 = 25$ except for $s = 23$ where there are two candidates; these candidates are depicted in Figure 7. The configurations are represented by matrices of size $2^d \times 3^d$ where the columns and the rows represent the cosets of $\Lambda/2$ and $\Lambda/3$ while the boxes represent the cosets of $\Lambda/6$. The various entries of the matrix give a complete description of the repartition of the cameras in the cosets of $\Lambda/2$, $\Lambda/3$ and $\Lambda/6$.

To decide between the candidates to optimality we are faced with the numerical evaluation of infinite products of the form $d'(k) = \prod_{p\neq 2,3}\left(1 - \frac{k}{p^d}\right)$ which converge very slowly (a power of $1/N$, if we take N terms). Using a technique developed by Vardi and Flajolet [6] efficient evaluation can be done.

3 Locating the Cameras

In the previous section we gave a description of the optimal configurations of size $s \leq 5^d$ in terms of relations, called p-visibility relations, and asserted the existence of such configurations. However, knowing such a family, $\mathcal{E}$ say, of relations, we still have to find a configuration of locations for the cameras that conform to these relations. This leads us to the concept of $\mathcal{E}$-configuration, which we discuss in the sequel.

Let $\mathcal{E} = (\sim_p)_{p\in\mathcal{P}}$ be a family of equivalence relations on the set $\mathcal{C} = \{C_1, \ldots, C_s\}$ of cameras. An s-tuple $(A_1, \ldots, A_s)$ of s integer lattice points is called an $\mathcal{E}$-*configuration* if

$$\text{for all } i, j,\ \ A_i = A_j \pmod p \iff C_i \sim_p C_j \tag{8}$$

is valid for all primes p. So an $\mathcal{E}$-configuration is a point in $\mathbb{Z}^{sd}$. The density of these points is given in the following theorem.

Theorem 11 *The density of the set of $\mathcal{E}$-configurations is given by the infinite product*

$$\prod_{p\in\mathcal{P}} \frac{(p^d)_{|\mathcal{C}/\sim_p|}}{p^{sd}} \tag{9}$$

where $(x)_y = x(x-1)(x-2)\ldots(x-y+1)$ is the descent factorial. Furthermore an $\mathcal{E}$-configuration exists if and only if the above density is non null.

Theorem 11 has some interesting consequences. It follows that an $\mathcal{E}$-configuration exists if and only if for p sufficiently large $\sim_p$ is the identity relation and for any prime p the cardinal of the quotient space $|\mathcal{C}/\sim_p|$ does not exceed p^d.

In addition, as long as we are concerned with an optimal configuration the above expression of the density of $\mathcal{E}$-configurations depends only on s. Indeed using condition (4) on optimal configurations the product (9) can be rewritten in the form

$$\prod_{k=1}^{s-1}\prod_{p\in\mathcal{P}}\left(1 - \frac{\min\{k, p^d - 1\}}{p^d}\right). \tag{10}$$

The size of expression (10) prohibits the use of unsophisticated random sampling in order to get $\mathcal{E}$-configurations. This is because the probability that a random s-camera configuration is an $\mathcal{E}$-configuration is $\leq 1/\zeta(d)^{s-1}$. Thus on the average it will be necessary to randomly sample at least $\zeta(d)^{s-1}$ times (which is exponential in s) before one succeeds in obtaining an $\mathcal{E}$-configuration. In [17] a simple randomized algorithm for doing this has expected time complexity $e^{O(s^{1/d})}$ (for d fixed). This raises the question of whether any iterative techniques starting from an arbitrary configuration will lead to an optimal one. Simulated annealing offers such an effective technique [21] but it is not known whether convergence to an optimal configuration can be achieved in polynomial time.

Returning to the original problem solved by Abbott, it will be interesting to determine the minimal size $(\max\{\|A_i - A_j\|\})$ of an optimal configuration and to determine the pattern of visible and non-visible points around the cameras.

4 Superposition of Lattices

Another interesting problem concerns generalizations of the camera placement problem to more general lattice systems, like tilings or even more generally of point configurations obtained by superposing lattices and/or tilings [8, 9].

Preliminary investigations [14] show that the problem reduces to the following three subproblems:

1. give a number theoretic characterization of the visibility relation among points of the given tiling system,
2. extend Rumsey's theorem; in particular, it is necessary to determine the density of the visibility sets $V(S)$ in arbitrary tiling systems,
3. investigate combinatorial optimization techniques in order to construct optimal configurations.

Item (1) is a nontrivial problem. Research in progress [14] shows that if the set of points is $\mathcal{O} = \Lambda \setminus \bigcup_{i=1}^{t} G_i$, where G_i are sublattices of Λ then for $x, y \in \mathcal{O}$ which are $\mathcal{O}$-visible (i.e. the line segment joining x and y avoids any points of $\mathcal{O}$) we must have that $\gcd(x-y) \leq 2^t$ (The proof uses a result of [3]. Moreover, determining whether or not $\mathcal{O}$ is empty is an NP-complete-problem [7].) For item (2) one requires proving stronger density theorems for visibility sets comparable to Rumsey's theorem [20]. For item (3) a reasonable approach is to refine and extend the algorithmic techniques and reduction theorems we have already developed for the case of $s \leq 5^d$ cameras. For more details see [14]. We illustrate the problems with an example.

Example 12 *Let the set of points be $\mathcal{O} = \{x \in \Lambda \mid x \neq 0 \pmod 3\}$. In that case a point x is visible from a point a if and only if $x \neq a \pmod{3, 5, 7, \ldots}$, and ($x \neq a \pmod 2$) or ($x = a \pmod 2$ and $x = -a \pmod 6$). We get then that the density of the set of points visible from each point of a finite subset S of $\mathcal{O}$ is given by the infinite product*

$$\frac{(2^d - |S/2|)(3^d - |S/3| - 1) + \omega}{6^d} \prod_{p>3} \left(1 - \frac{|S/p|}{p^d}\right)$$

where ω is the cardinality of the set of cosets c of $\mathcal{O}/6$ such that

$$\forall a \in S \quad c \neq a \pmod 3$$
$$\exists a \in S \quad c = a \pmod 2$$
$$\forall a \in S \quad c = a \pmod 2 \Rightarrow c = -a \pmod 6.$$

What can we say for the camera placement problem? Much of the prev optimization analysis holds in this case as well. We can show that

- *the optimality condition (4) holds for $p = 3, 5, \ldots$,*
- *for the s-camera placement problem ($s < 5^d$) the optimization prob becomes*

$$\max \left\{ \sum_{I \in \mathcal{O}/6} u'(b_I) \mid b_I = |I \cap (-S)| + \sum_{d(I,J)=2} |J \cap S| \right\},$$

  where u' is the reduced density function.

Then we can give a complete characterization of the optimal configuration size $s \leq 2^d$; for example an optimal 2-camera configuration $\{a, b\}$ is character by a and b are visible in Λ and $a + b = 0 \pmod 3$.

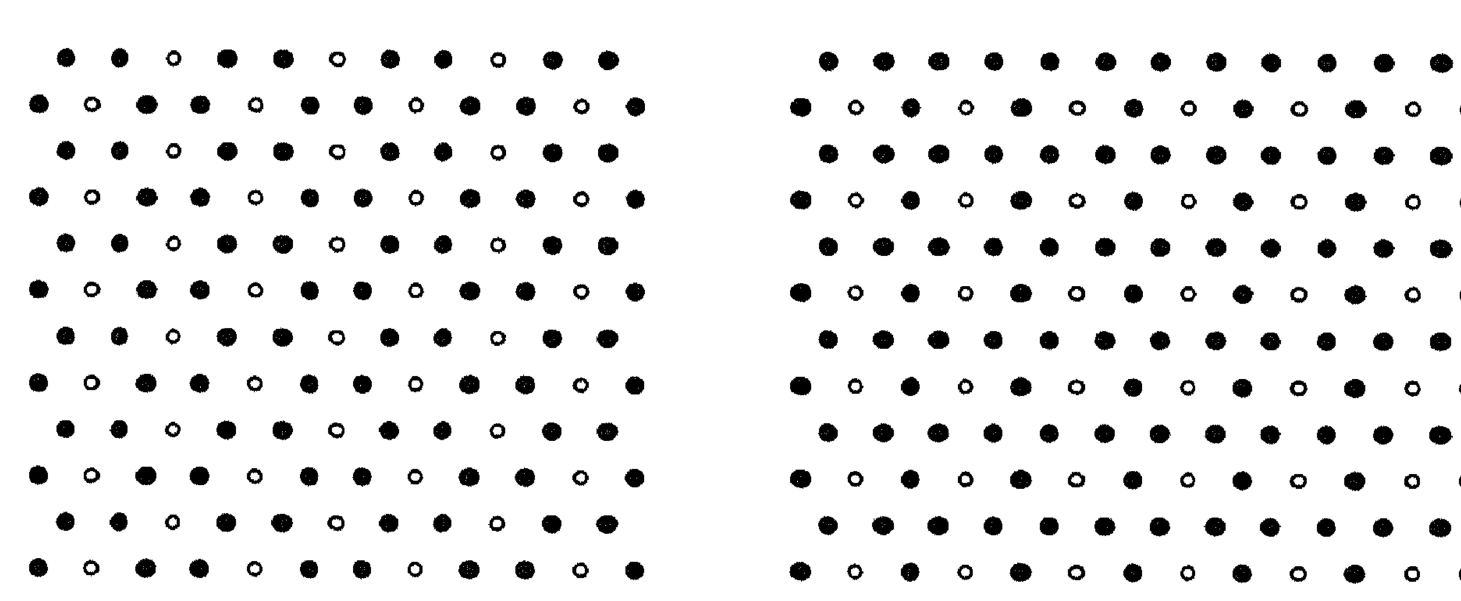

FIGURE 8. The Honeycomb (left), Hexagonal (right) Lattices

Another interesting class of visibility problems is also obtained when we sume that the set of "obstacle" points (i.e. which block the visibility) and set of possible locations of the cameras make a partition of Λ. Example such tiling systems are depicted in Figure 8 where we assume that the • are obstacle-points and the ∘ are the possible locations of the cameras. We ment the following example illustrating this problem.

EXAMPLE 13 *Let F be a subgroup of Λ/n, where $n = p_1 \ldots p_r$ is free of squ and let F' be its complement. Let the set of obstacles be*

$$\mathcal{O} = \{ x \in \Lambda \mid \exists f \in F' \, (x = f \pmod n) \}$$

and the set of possible locations of the guards be

$$\mathcal{G} = \{ x \in \Lambda \mid \exists f \in F \, (x = f \pmod{n}) \} .$$

In this instance it can be shown that two lattice points x, y are visible if and only if they are p-visible for all primes $p \neq p_1, \ldots, p_r$. Moreover the density of obstacles visible from a finite set S of guards is given by the infinite product

$$\prod_{\substack{p \neq p_i \\ p \in \mathcal{P}}} \left(1 - \frac{|S/p|}{p^d} \right) .$$

The optimization analysis developed for the camera placement problem in Λ holds here as well, except that we have to "forget" completely the p-visiblity relation for $p = p_1, \ldots, p_r$. This allows us to control the value of the reduced density function and give some insight on the validity of our conjecture (minimal variance) about optimal configurations (for more details see [17]).

5 Conclusion

In the previous sections we analyzed the camera placement problem in complete integer lattices. We have shown how to reduce the problem to a non-linear integer optimization problem and used our analysis to study optimal s-camera configurations when $s \leq 5^d$. The main challenge in this area is to derive characterizations of optimal configurations for $s > 5^d$ cameras that lead to a polynomial number of candidates to optimality.

We also considered the more general case of lattice systems arising from the superposition of complete integer lattices, as well as lattice configurations arising from specific sets of obstacle and cameras. In all these cases we have the additional tasks of providing number theoretic characterizations of visibility as well as new density theorems. However, our optimization methodology is fairly general and will be useful even in these more general cases.

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